EXERCISE – V JEE PROBLEMS

1. (a) If $i=\sqrt{-1}$, then $4+5\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{334}+3\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{365}$ is equal to **[JEE 99, 2+10]**

(A) $1-i\sqrt{3}$ (B) $-1+i\sqrt{3}$ (C) $i\sqrt{3}$ (D) $-i\sqrt{3}$

(b) For complex numbers z & ω, prove that,

$|z|^2\omega-|\omega|^2 z=z-\omega$ if and only if, $z=\omega$ or $z\overline{\omega}=1$

2. (i) If $\alpha=e^{\frac{2\pi i}{7}}$ and $f(x)=A_0+\sum_{k=1}^{20}A_k x^k$, then find the value of, $f(x)+f(\alpha x)+.......+f(\alpha^6 x)$ independent of α. **[REE 99, 6+3]**

(ii) Let $\alpha+i\beta$; $\alpha,\beta\in R$, be a root of the equation $x^3+qx+r=0$,; $q, r\in R$. Find a real cubic equation, independent of α & β, whose one root is 2α.

3. (a) If z_1, z_2, z_3 are complex number such that $|z_1|=|z_2|=|z_3|=\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|=1$, then $|z_1+z_2+z_3|$ is **[JEE 2000 (Scr.), 1+1]**

(A) equal to 1 (B) less than 1
(C) greater than 3 (D) equal to 3

(b) If $\arg(z)<0$, then $\arg(-z)-\arg(z)$ equals

(A) π (B) $-\pi$ (C) $-\frac{\pi}{2}$ (D) $\frac{\pi}{2}$

4. Given, $z=\cos\frac{2\pi}{2n+1}+i\sin\frac{2\pi}{2n+1}$, 'n' a positive integer, find the equation whose roots are,
$\alpha=z+z^3+....+z^{2n-1}$ & $\beta=z^2+z^4+.......+z^{2n}$. **[REE 2000 (Mains), 3]**

5. Find all those roots of the equation $z^{12}-56z^6-512=0$ whose imaginary part is positive. **[REE 2000, 3]**

6. (a) The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1-z_3}{z_2-z_3}=\frac{1-i\sqrt{3}}{2}$ are the vertices of a triangle which is **[JEE 2001 (Scr.), 1 + 1]**

(A) of area zero (B) right angled isosceles
(C) equilateral (D) obtuse angled isosceles

(b) Let z_1 and z_2 be the n^{th} roots of unity which subtend a right angle at the origin. Then n must be of the form

(A) 4k + 1 (B) 4k + 2 (C) 4k + 3 (D) 4k

7. (a) Let $\omega=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$. Then the value of the sdeterminant $\begin{vmatrix}1 & 1 & 1\\ 1 & -1-\omega^2 & \omega^2\\ 1 & \omega^2 & \omega^4\end{vmatrix}$ is **[JEE 2002 (Scr.) 3+3]**

(A) 3ω (B) $3\omega(\omega-1)$ (C) $3\omega^2$ (D) $3\omega(1-\omega)$

(b) For all complex numbers z_1, z_2 satisfying $|z_1|=12$ and $|z_2-3-4i|=5$, the minimum value of $|z_1-z_2|$ is

(A) 0 (B) 2 (C) 7 (D) 17

(c) Let a complex number α, $\alpha\neq 1$, be a root of the equation $z^{p+q}-z^p-z^q+1=0$, where p, q are distinct primes. Show that either $1+\alpha+\alpha^2+........+\alpha^{p-1}=0$ or $1+\alpha+\alpha^2+.......+\alpha^{q-1}=0$, but not both together. **[JEE 2002, 5]**

8. (a) If z_1 and z_2 are two complex numbers such that $|z_1|<1<|z_2|$ then prove that $\left|\frac{1-z_1\bar{z}_2}{z_1-z_2}\right|<1$.

[JEE 2003, 2+2]

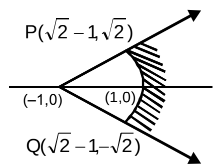
(b) Prove that there exists no complex number z such that $|z|<1/3$ & $\sum_{r=1}^{n}a_r z^r=1$ where $|a_r|<2$.

9. (a) ω is an imaginary cube root of unity. If $(1 + \omega^2)^m = (1 + \omega^4)^m$, then the least positive integral value of m is **[JEE 2004 (Scr.)]**

(A) 6 (B) 5 (C) 4 (D) 3

(b) Find the centre and radius of circle determined by all complex numbers z = x + iy satisfying $\left|\frac{z-\alpha}{z-\beta}\right| = k$, where $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$ are fixed complex and $k \neq 1$ **[JEE 2004, 2]**

10. (a) The locus of z which lies in shaded region (excluding the boundaries) is best represented by **[JEE 2005 (Scr.), 3+3]**



(A) z : |z + 1| > 2 and |arg (z + 1)| < $\pi/4$

(B) z : |z – 1| > 2 and |arg (z – 1)| < $\pi/4$

(C) z : |z + 1| < 2 and |arg (z + 1)| < $\pi/2$

(D) z : |z – 1| < 2 and |arg (z – 1)| < $\pi/2$

(b) If a, b, c are integers not all equal and ω is cube root of unity ($w \neq 1$), then the minimum value of $|a + b\omega + c\omega^2|$ is

(A) 0 (B) 1 (C) $\sqrt{3}/2$ (D) 1/2

(c) If one of the vertices of the square circumscribing the circle $|z - 1| = \sqrt{2}$ is $2 + \sqrt{3}$ i. Find the other vertices of the square. **[JEE 2005 (Mains), 4]**

11. If $w = \alpha + i\beta$ where $\beta \neq 0$ and $z \neq 1$, satisfies the condition that $\frac{w - \bar{w}z}{1-z}$ is purely real , then the set of the values of z is **[JEE 2006, 3]**

(A) {z : |z| = 1} (B) $\{z : z = \bar{z}\}$

(C) $\{z : z \neq 1\}$ (D) $\{z : |z| = 1, z \neq 1\}$

12. (a) A man walks a distance of 3 units from the origin towards the north-east (N 45° E) direction. Form there, he walks a distance of 4 units towards the north-west (N 45° W) direction to reach a point P. Then the position of P in the Argand plane is **[JEE 2007, 3+3]**

(A) $3e^{i\pi/4} + 4i$ (B) $(3 - 4i)e^{i\pi/4}$

(C) $(4 + 3i)e^{i\pi/4}$ (D) $(3 + 4i)e^{i\pi/4}$

(b) If |z| = 1 and $z \neq \pm 1$, then all the values of $\frac{z}{1-z^2}$ lie on

(A) a line not passing through the origin

(B) $|z| = \sqrt{2}$ (C) the x-axis (D) the y-axis

13. (a) A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i} + \hat{j}$ and then it moves through an angle $\frac{\pi}{2}$ in anticlockwise direction on a circle with centre at origin, to reach a point z_2. The point z_2 is given by **[JEE 2008, 3+4+4+4]**

(A) 6 + 7i (B) –7 + 6i (C) 7 + 6i (D) –6 + 7i

(b) Comprehension (3 questions together)

Let A, B, C be three sets of complex numbers as defined below

A = {z : Im $z \geq 1$}

B = {z : |z – 2 – i| = 3}

C = {z : Re((1 – i) z) = $\sqrt{2}$ }.

(i) The number of elements in the set $A \cap B \cap C$ is

(A) 0 (B) 1 (C) 2 (D) ∞

(ii) Let z be any point in $A \cap B \cap C$. Then $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

(A) 25 & 29 (B) 30 & 34 (C) 35 & 39 (D) 40 & 44

(iii) Let z be any point in $A \cap B \cap C$ and let w be any point satisfying |w – 2 – i| < 3.

Then, |z| – |w| + 3 lies between

(A) –6 & 3 (B) –3 & 6 (C) –6 & 6 (D) –3 & 9

14. Let z = x + iy be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $\bar{z}z^3 + z\bar{z}^3 = 350$ is **[JEE 2009]**

(A) 48 (B) 32 (C) 40 (D) 80

15. Let $z = \cos\theta + i\sin\theta$. Then the value of $\sum_{m=1}^{15} \text{Im}(z^{2m-1})$ at $\theta = 2^\circ$ is **[JEE 2009]**

(A) $1/\sin 2^\circ$ (B) $1/3\sin 2^\circ$
(C) $1/2\sin 2^\circ$ (D) $1/4\sin 2^\circ$

16. Let p and q be real numbers such that $p \neq 0$, $p^3 \neq q$ and $p^3 \neq -q$. If α and β are nonzero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is **[JEE 2010]**

(A) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$
(B) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$
(C) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$
(D) $(p^3 - q)x^2 + (5p^3 + 2q)x + (p^3 - q) = 0$

17. Let ω be a complex cube root of unity with $\omega \neq 1$. A fair die is thrown three times. If r_1, r_2 and r_3 are the numbers obtained on the die, then the probability that $\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$ is **[JEE 2010]**

(A) $\frac{1}{18}$ (B) $\frac{1}{9}$ (C) $\frac{2}{9}$ (D) $\frac{1}{36}$

18. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg(w) denotes the principal argument of a nonzero complex number w, then **[JEE 2010]**

(A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$
(B) $\text{Arg}(z - z_1) = \text{Arg}(z - z_2)$
(C) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$
(D) $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$

19. Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$ is equal to **[JEE 2010]**

20. Match the statement in Column I with those in Column II.
[Note : Here z takes values in the complex plane and Im z and Re z denote, respectively, the imaginary part and the real part of z]. **[JEE 2010]**

Column–I	**Column–II**
(A) The set of points z satisfying $\lvert z - i\lvert z\rvert\rvert = \lvert z + i\lvert z\rvert\rvert$ is contained in or equal to	(P) an ellipse with eccentricity $\frac{4}{5}$
(B) The set of points z satisfying $\lvert z + 4\rvert + \lvert z - 4\rvert = 10$ is contained in or equal to	(Q) the set of points z satisfying Im z = 0
(C) If $\lvert w\rvert = 2$, then the set of points $z = w - \frac{1}{w}$ is contained in or equal to	(R) the set of points z satisfying $\lvert \text{Im } z\rvert \leq 1$
(D) If $\lvert w\rvert = 1$, then the set of points $z = w + \frac{1}{w}$ is contained in or equal to	(T) the set of points z satisfying $\lvert z\rvert \leq 3$

Paragraph for Question Nos. 21 to 23

Let a, b and c be three real numbers satisfying

$$[a\ \ b\ \ c]\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0\ \ 0\ \ 0] \quad .....(E)$$ **[JEE 2011]**

21. If the point P(a, b, c), with reference to (E), lies on the plane $2x + y + z = 1$, then the value of $7a + b + c$ is

(A) 0 (B) 12 (C) 7 (D) 6

22. Let ω be a solution of $x^3 - 1 = 0$ with Im $(\omega) > 0$. If $a = 2$ with b and c satisfying (E), then the value of $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$ is equal to

(A) –2 (B) 2 (C) 3 (D) –3

23. Let $b = 6$, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n$ is

(A) 6 (B) 7 (C) 6/7 (D) ∞

24. If z is any complex number satisfying $|z-3-2i| \le 2$, then the minimum value of $|2z - 6 + 5i|$ is **[JEE 2011]**

25. Let $\omega \neq 1$ be cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$

where each of a, b and c is either ω or ω^2. Then the number of distinct matrices in the set S is **[JEE 2011]**

(A) 2 (B) 6 (C) 4 (D) 8

26. Let $\omega = e^{i\pi/3}$, and a, b, c, x, y, z be non-zero complex numbers such that

$$a + b + c = x$$
$$a + b\omega + c\omega^2 = y$$
$$a + b\omega^2 + c\omega = z$$

Then the value of $\dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is **[JEE 2011]**

27. Match the statements given in **Column I** with the values given in **Column II** **[JEE 2011]**

Column – I **Column – II**

(A) If $\vec{a} = \hat{j} + \sqrt{3}\hat{k}, \vec{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is (P) $\dfrac{\pi}{6}$

(B) If $\int_a^b (f(x) - 3x)\,dx = a^2 - b^2$, then the value of $f\left(\dfrac{\pi}{6}\right)$ is (Q) $\dfrac{2\pi}{3}$

(C) The value of $\dfrac{\pi^2}{\ln 3} \displaystyle\int_{7/6}^{5/6} (\sec(\pi x)\,dx$ is (R) $\dfrac{\pi}{3}$

(D) The maximum value of $\left|\text{Arg}\left(\dfrac{1}{1-z}\right)\right|$ for $|z| = 1, z \neq 1$ is given by (S) π

(T) $\dfrac{\pi}{2}$

28. Match the statements given in **Column I** with the intervals/union of intervals given in **Column II** **[JEE 2011]**

Column – I **Column – II**

(A) The $\left\{\text{Re}\left(\dfrac{2iz}{1-z^2}\right) : z \text{ is a complex number}, |z| = 1, z \neq \pm 1\right\}$ is (P) $(-\infty, -1) \cup (1, \infty)$

(B) The domain of the function $f(x) = \sin^{-1}\left(\dfrac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is (Q) $(-\infty, 0) \cup (0, \infty)$

(C) If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta) : 0 \le \theta < \dfrac{\pi}{2}\right\}$ is (R) $[2, \infty)$

(S) $(-\infty, -1] \cup [1, \infty)$

(D) If $f(x) = x^{3/2}(3x-10), x \ge 0$, then f(x) is increasing in (T) $(-\infty, 0] \cup [2, \infty)$

29. Let z be a complex number such that the imaginary part of z is nonzero and $a = z^2 + z + 1$ is real. Then a **cannot** take the value **[JEE 2012]**

(A) -1 (B) $\dfrac{1}{3}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{3}{4}$